JOURNAL OF COMPUTATIONAL PHYSICS 2, 339–355 (1968)

Heuristic Stability Theory for Finite-Difference Equations*

C. W. HIRT

Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico 87544

ABSTRACT

A simple method is proposed for investigating the computational stability of finite-difference equations. The technique is especially powerful because of its applicability to nonlinear equations with variable coefficients. The method, which is based on an examination of certain kinds of truncation errors, is illustrated by applying it to a simple linear difference equation. Then it is used to explain the origin of instabilities observed in calculations of one- and two-dimensional fluid flows.

I. INTRODUCTION

Attempts to solve partial differential equations by finite-difference methods often end in disaster. A finite-difference equation may have a rapidly growing and oscillating solution that in no way resembles the solution expected from the differential equation. In such a situation the difference equation is said to be computationally unstable. Obviously, it is desirable to avoid these disasters. For linear difference equations with constant coefficients, stability can be determined using a Fourier method proposed by von Neuman [1]. Unfortunately, most equations of physical interest are either nonlinear, or have nonconstant coefficients, or both. In this paper a simple heuristic method is proposed for investigating the computational stability of such finite difference equations.

The discussion given here is not rigorous or complete, but presents a technique that we have found extremely useful. It is based on a rather simple idea. A finite-difference equation is reduced to a differential equation by expanding each of its terms in a Taylor series. The lowest-order terms in the expansion must represent the approximated differential equation. All higher-order terms are called truncation errors. This paper shows that the stability of a difference equation can often be determined from an examination of these truncation errors.

The technique is illustrated by applying it to several examples. First, a linear

* This work was performed under the auspices of the United States Atomic Energy Commission.

581/2/4-1

difference equation is discussed. The Fourier stability analysis [1] and the truncation-error method can be directly compared in this case. The example reveals two basic kinds of instability.

Second, the technique is applied to the coupled, nonlinear equations of motion for a one-dimensional compressible fluid. Four different finite-difference approximations are used in test calculations. Computational instabilities are observed that are not predicted by a linear Fourier stability analysis. The source of these instabilities is easily found using the technique proposed here.

Finally, the method is used to explain an instability observed in an application of the Marker-And-Cell (MAC) method for computing the dynamics of an incompressible fluid [2].

II. Linear Example

The connection between truncation errors and computational stability is easy to determine for linear equations. Take, for example, the equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

which describes the convection and diffusion of a function $u(x, t)$. The convection velocity c and diffusion coefficient ν are assumed constant. Solutions of this equation are bounded and otherwise well-behaved.

A simple finite difference approximation to (1) is

$$\frac{u_j^{n+1} - u_j^n}{\delta t} = -\frac{c}{2\delta x}(u_{j+1}^n - u_{j-1}^n) + \frac{\nu}{\delta x^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n), \tag{2}$$

where, e.g., u_j^n denotes $u(j\delta x, n\delta t)$.

Observe that the difference equation (2) propagates an effect from each datum point $(j\delta x, n\delta t)$ into a region bounded by lines passing through that point and having slopes $\pm\delta x/\delta t$. These lines are not true characteristic lines along which signals propagate, but they play an important role in our subsequent stability discussion.

The difference equation (2) is linear with constant coefficients, hence, its stability can be examined by the Fourier method. A single Fourier component of u_j^n, say

$$r^n \exp(ikj\delta x), \tag{3}$$

satisfies (2) provided

$$r = 1 - \left(\frac{ic\delta t}{\delta x}\right)\sin(k\delta x) - \left(\frac{2\nu\delta t}{\delta x^2}\right)[1 - \cos(k\delta x)]. \tag{4}$$

If r has a magnitude greater than unity for any value of k, then the difference equation is unstable, since that Fourier component will grow exponentially with n, i.e., with time. A study of (4) shows that the magnitude of r is less than unity for all k if two conditions are satisfied,

$$\frac{2\nu\delta t}{\delta x^2} \leqslant 1, \qquad \nu \geqslant \frac{1}{2} c^2 \delta t. \tag{5}$$

These two inequalities are stability conditions for (2). For given values of ν, c, and δx they define a range of permissible δt values.

In the remainder of this section we wish to show that the stability conditions (5) can also be obtained from an examination of truncation errors. For this purpose, consider each term in (2) as a continuous function of x and t (see Ref. [8]). For example, consider u_{j+1}^n as denoting $u(x + \delta x, t)$. Expand each term of (2) in a Taylor series about the point (x, t) to obtain

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = -\frac{1}{2} \delta t \frac{\partial^2 u}{\partial t^2} + O(\delta x^2, \delta t^2). \tag{6}$$

All second- and higher-order terms in δx and δt are represented by the order symbol $O(\delta x^2, \delta t^2)$. The zero-order terms on the left-hand-side of (6) form the original differential equation (1). This is consistent with the requirement that the approximation improve as δx and δt tend to zero.

If the Fourier component $\exp[i(kx + \omega t)]$ is substituted into (6), the result is equal to the r Eq. (4), with $x = j\delta x$, $r = \exp(i\omega\delta t)$, and with all sines, cosines, and r expanded in powers of $k\delta x$ and $\omega\delta t$. This suggests a useful approximation to (6). The lowest-order even and odd derivative terms in (6), with respect to each independent variable, generate the corresponding lowest-order real and imaginary terms in the expanded form of (4). Since we are primarily interested in solutions of the difference equation with wavelengths greater than or equal to $2\delta x$, or with wave numbers $k\delta x$ in the interval $(0, \pi)$, the lowest-order terms in (4) serve as a useful approximation. Thus, (6) is approximated by keeping only the lowest-order even and odd derivative terms, which are

$$\frac{\delta t}{2} \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0. \tag{7}$$

It is significant that (7) is not identical to (1), which we set out to approximate. In fact, it is the difference between these equations that accounts for the computational instabilities of (2).

To see this, recall that the difference equation propagates information into a region bounded by lines whose slopes are $dx/dt = \pm\delta x/\delta t$. The δt term in (7) makes that equation hyperbolic, with characteristic lines whose slopes are $dx/dt = \pm(2\nu/\delta t)^{1/2}$. If the difference equation is to have solutions approximating those of (7), then its region of influence must include the region of influence of (7). The necessary condition is

$$\frac{2\nu}{\delta t} \leqslant \left(\frac{\delta x}{\delta t}\right)^2. \tag{8}$$

Courant, Friedrichs, and Lewy [3] used a similar region of influence condition for finite-difference approximations to linear wave equations. Their condition, now called the Courant condition, restricts the distance a wave travels in one time increment to less than one space interval. A violation of the Courant condition leads to an oscillating and exponentially growing instability. Apparently this is also true in our case, for condition (8) is identical to the first stability condition (5), and a violation of condition (8) allows (2) to have exponentially growing solutions that oscillate with increasing n.

The second stability condition can also be obtained from (7), if the term proportional to δt is rewritten. From (6) we find that

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - 2\nu c \frac{\partial^3 u}{\partial x^3} + \nu^2 \frac{\partial^4 u}{\partial x^4} + O(\delta t). \tag{9}$$

Combining (9) with (7), and neglecting terms of second order in δt,

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = \left(\nu - \frac{1}{2} c^2 \delta t\right) \frac{\partial^2 u}{\partial x^2} + \nu c \delta t \frac{\partial^3 u}{\partial x^3} - \frac{\delta t}{2} \nu^2 \frac{\partial^4 u}{\partial x^4}. \tag{10}$$

This is identical to the result that would have been obtained from a Taylor expansion of (2) about the point $x = j\delta x$, $t = (n + \frac{1}{2})\,\delta t$. The last two terms in (10) can be dropped, as before, since they are higher-order derivatives, so that

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = \left(\nu - \frac{1}{2} c^2 \delta t\right) \frac{\partial^2 u}{\partial x^2}. \tag{11}$$

Comparing this result with (1) we now find there is an additional diffusion term. When (11) has a negative diffusion coefficient it has solutions that grow exponentially in time. For nongrowing, stable solutions, it is necessary that

$$\nu \geqslant \tfrac{1}{2} c^2 \delta t. \tag{12}$$

This is also the difference equation stability condition (5). When (12) is violated the difference equation has exponentially growing solutions.

These results are easily summarized. The stability conditions for the linear difference equation have been obtained from the differential equations (7), (11). These approximate equations were obtained from Taylor series expansions of the difference equation by keeping only the lowest-order even and odd derivative terms with respect to each independent variable. The stability condition (8) was a region-of-influence condition for Eq. (7). Violation of this condition results in *oscillating* and exponentially growing solutions. Condition (12) guarantees a positive diffusion coefficient in (11). Violation of this condition results in *pure* exponentially growing solutions.

In the remainder of this paper generalizations of these conditions are applied to difference equations that are nonlinear and have nonconstant coefficients.

III. Nonlinear Example

The equations of motion for a fluid are coupled, nonlinear, partial differential equations. Finite-difference approximations for these equations are easily constructed, but they frequently suffer from computational instabilities. The usual Fourier method of testing for stability refers to perturbations about stationary and uniform flows. In this section, the truncation error method is extended to study instabilities that cannot be predicted by the Fourier method.

The Eulerian equations of motion for a fluid in one space dimension are

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) &= 0, \\ \frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p + q) &= 0, \qquad (13) \\ \frac{\partial \rho(I + \frac{1}{2}u^2)}{\partial t} + \frac{\partial}{\partial x}\left\{\rho\left[I + \frac{1}{2}u^2\right]u + (p+q)\,u\right\} &= 0, \end{aligned}$$

where ρ is the fluid density, u the velocity, p the fluid pressure, and I the specific internal energy. An artifical viscosity term, q, has been added to permit the numerical calculation of shock waves [4]. The form we have chosen for q, which is different from that suggested in [4], is

$$q = \begin{cases} -\alpha \delta x \rho \dfrac{\partial u}{\partial x}, & \text{if} \quad \dfrac{\partial u}{\partial x} < 0, \\ 0 \quad , & \text{if} \quad \dfrac{\partial u}{\partial x} \geqslant 0. \end{cases} \qquad (14)$$

The parameter α is a constant having the dimensions of a velocity. Equations (13) are written in conservative form, i.e., a time derivative plus the divergence of a flux.

A stable and conservative finite-difference approximation can be constructed from Eq. (13) by using the method of donor-cell differencing for the convection terms [5]. The difference equations are

$$
\begin{aligned}
\rho_j^{n+1} &= \rho_j^n - \frac{\delta t}{\delta x}\,[\langle\rho\rangle_{j+1/2}^n u_{j+1/2}^n - \langle\rho\rangle_{j-1/2}^n u_{j-1/2}^n],\\
(\rho u)_j^{n+1} &= (\rho u)_j^n - \frac{\delta t}{\delta x}\,[\langle\rho u\rangle_{j+1/2}^n u_{j+1/2}^n - \langle\rho u\rangle_{j-1/2}^n u_{j-1/2}^n\\
&\qquad + p_{j+1/2}^n + q_{j+1/2}^n - p_{j-1/2}^n - q_{j-1/2}^n],\\
[\rho(I+\tfrac{1}{2}u^2)]_j^{n+1} &= [\rho(I+\tfrac{1}{2}u^2)]_j^n - \frac{\delta t}{\delta x}\,[\langle\rho(I+\tfrac{1}{2}u^2)\rangle_{j+1/2}^n u_{j+1/2}^n\\
&\qquad - \langle\rho(I+\tfrac{1}{2}u^2)\rangle_{j-1/2}^n u_{j-1/2}^n + (p+q)_{j+1/2}^n u_{j+1/2}^n\\
&\qquad - (p+q)_{j-1/2}^n u_{j-1/2}^n].
\end{aligned}
\tag{15}
$$

Quantities evaluated on boundaries, $x = (j \pm \frac{1}{2})\,\delta x$, are taken as simple averages, e.g.,

$$
u_{j+1/2}^n = \tfrac{1}{2}(u_{j+1}^n + u_j^n). \tag{16}
$$

Donor-cell boundary values for a quantity Q are denoted by $\langle Q\rangle$ and are defined by

$$
\langle Q\rangle_{j+1/2}^n = \begin{cases} Q_j^n, & \text{if} \quad u_{j+1/2}^n \geqslant 0,\\ Q_{j+1}^n, & \text{if} \quad u_{j+1/2}^n < 0. \end{cases} \tag{17}
$$

Difference equations (15) are known to be stable in a wide variety of applications. In the following we study the stability of Eq. (15) when alternative difference approximations are substituted for the mass convection term, but the other equations are left intact. The cases considered are, in terms of $m_j^n \equiv \rho_j^n u_j^n$,

(a) donor cell, same as Eq. (15),

$$
\rho_j^{n+1} = \rho_j^n - \frac{\delta t}{\delta x}\,[\langle\rho\rangle_{j+1/2}^n u_{j+1/2}^n - \langle\rho\rangle_{j-1/2}^n u_{j-1/2}^n],
$$

(b) centered difference

$$
\rho_j^{n+1} = \rho_j^n - \frac{\delta t}{2\delta x}\,[m_{j+1}^n - m_{j-1}^n], \tag{18}
$$

(c) centered difference advanced in time

$$\rho_j^{n+1} = \rho_j^{\,n} - \frac{\delta t}{2\delta x}\,[m_{j+1}^{n+1} - m_{j-1}^{n+1}],$$

and

(d) second-order donor cell

$$\rho_j^{n+1} = \rho_j^{\,n} - \frac{\delta t}{\delta x}\,[\langle\!\langle m\rangle\!\rangle_{j+1/2}^{n+1} - \langle\!\langle m\rangle\!\rangle_{j-1/2}^{n+1}],$$

where

$$\langle\!\langle m\rangle\!\rangle_{j+1/2}^{n+1} = \begin{cases} m_j^{n+1} + \frac{1}{4}\,(m_{j+1}^{n+1} - m_{j-1}^{n+1}), & \text{if} \quad m_{j+1/2}^{n+1} \geqslant 0 \\ m_{j+1}^{n+1} - \frac{1}{4}\,(m_{j+2}^{n+1} - m_j^{n+1}), & \text{if} \quad m_{j+1/2}^{n+1} < 0. \end{cases}$$

In all cases except (b) the difference equations (15), (18) are stable according to a linearized Fourier stability analysis. Actual calculations, however, exhibit computational instabilities in cases (b) thru (d). By developing the ideas presented in Section II the origin of these instabilities is easily located.

A broken diaphragm (shock tube) problem was chosen to test the difference equations. The setup consisted of a tube with closed ends and divided in half by a diaphragm. Both halves initially contain gas at rest, but the left half is at a higher pressure than the right half. Numerical calculations are started at the time when the diaphragm is broken. At a later time the system consists of a shock moving to the right, followed by a contact surface also moving to the right, and a rarefaction moving to the left.

The density profile for a typical broken diaphragm calculation, obtained with the stable donor-cell method (a), is shown in Fig. 1. The corresponding velocity profile is shown in Fig. 2. For this calculation a polytropic equation of state was used with a ratio of specific heats equal to 5/3, an initial pressure and density ratio of 5, $\delta t = 0.25$, $\delta x = 1.0$, and $\alpha = 1.0$. The shock, contact surface, and rarefaction corners are smoothed out over several computational cells, because of artificial viscosity and donor-cell effects.

This same calculation was repeated for each variation of the mass equation (18b)–(18d). Results of these variations are shown in Figs. 3–5. In each case there is an instability in the region behind the contact surface. These instabilities do not flip-flop on successive computational cycles, which suggests they are related to a lack of mass diffusion.

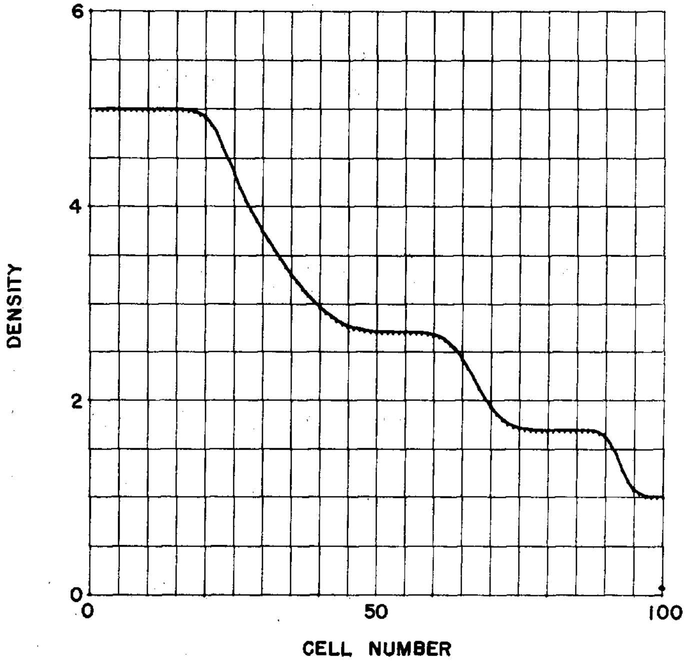

FIG. 1. A plot of density vs cell number at $t = 30.00$, as calculated by the stable donor-cell method (18a).

To test this idea, we need the truncation errors associated with each mass equation (18a)–(18d). However, only those terms that involve a second space derivative of ρ are needed, since they are the only ones that contribute to a mass diffusion. The effective diffusion coefficients for each case in (18), through terms of order δt and δx^4 are

$$
\begin{aligned}
&\text{(a)} \quad \frac{\delta x}{2} | u | - \frac{\delta t}{2} (u^2 + c^2) - \frac{\delta x^2}{4} \frac{\partial u}{\partial x} + \frac{\delta x^3}{8} \frac{\partial^2 u}{\partial x^2} - \frac{\delta x^4}{24} \frac{\partial^3 u}{\partial x^3}, \\
&\text{(b)} \quad - \frac{\delta t}{2} (u^2 + c^2) - \frac{\delta x^2}{2} \frac{\partial u}{\partial x} - \frac{\delta x^4}{12} \frac{\partial^3 u}{\partial x^3}, \\
&\text{(c)} \quad + \frac{\delta t}{2} (u^2 + c^2) - \frac{\delta x^2}{2} \frac{\partial u}{\partial x} - \frac{\delta x^4}{12} \frac{\partial^3 u}{\partial x^3}, \\
&\text{(d)} \quad + \frac{\delta t}{2} (u^2 + c^2) + \frac{\delta x^2}{4} \frac{\partial u}{\partial x} - \frac{3}{4} \delta x^3 \frac{\partial^2 u}{\partial x^2} + \frac{13 \delta x^4}{24} \frac{\partial^3 u}{\partial x^3},
\end{aligned}
\qquad (19)
$$

where c denotes the adiabatic speed of sound. Space truncation errors have been retained to a higher order than time errors, because they are enhanced by the initial discontinuity.

The results of Section II indicate that instabilities can occur wherever a diffusion coefficient is negative. We shall show that this is also true here. For a short time after the breaking of the diaphragm, the effective diffusion coefficients (19) are negative in certain regions, because of the large initial velocity gradients. At late times, all coefficients (19) are positive if evaluated from a mean velocity profile, but the instabilities persist. This indicates that nonlinear effects arising from the instabilities themselves are influencing the results. However, without directly evaluating the coefficients in (19), we can see the influence on stability of individual truncation terms by comparing results from cases (18a)–(18d).

The donor-cell terms in (19a) do not appear in (19b), making the leading term in (19b) proportional to δt and negative. This δt term is the only contribution to (19b) in the constant velocity region between the rarefaction foot and the shock. Hence, the method is unstable, see Fig. 3. There is an instability starting at the

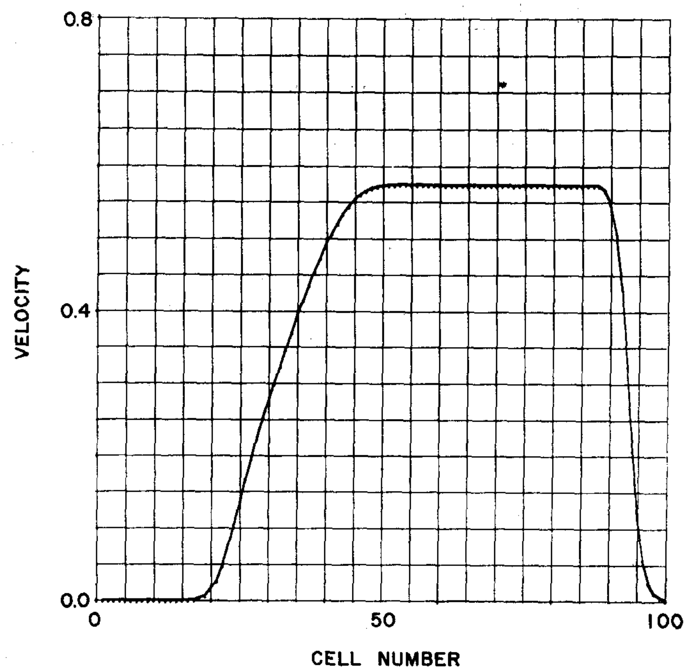

FIG. 2. A plot of velocity vs cell number at t = 30.00, as calculated by the stable donor-cell method (18a).

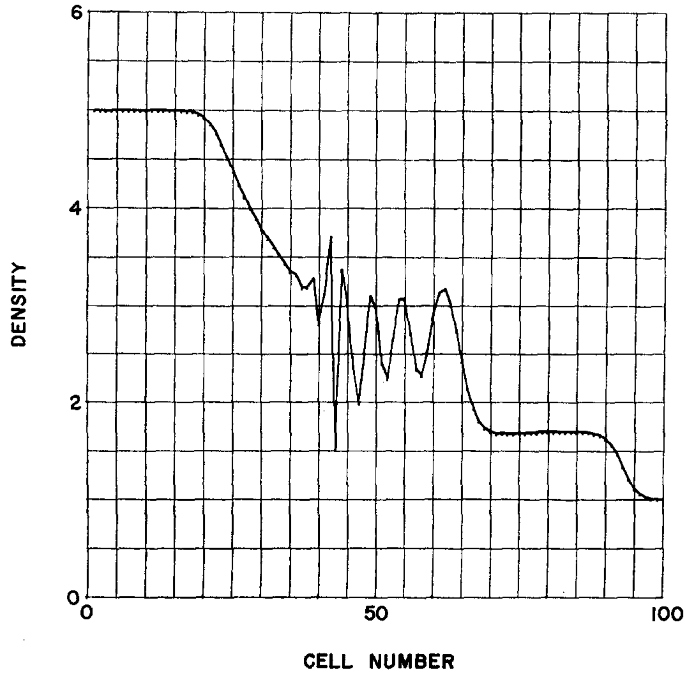

FIG. 3. A plot of density vs cell number at $t = 30.00$, as calculated by the centered-difference method (18b).

foot of the rarefaction and continuiting into the constant-velocity region, up to the contact surface. There is no corresponding instability in the constant-velocity region between the contact surface and the shock. This is probably associated with the absence of a perturbation, since fluid enters that region by passing through the shock where it is smoothed by artificial viscosity effects.

Case (18c) has the same diffusion coefficient as (18b), except the δt term is positive. If the instability arising in (18b) is solely an effect of the δt term, then (18c) should be stable. This is not the case. In Fig. 4 an instability is observed much like that in Fig. 3. The amplitude of the instability is decreasing in the constant-velocity region where only the positive δt diffusion term is acting, but further inspection of Fig. 4 reveals that the instability starts in the rarefaction at the point where the $\partial^3 u/\partial x^3$ term in (19c) turns negative. A corresponding instability is not observed at the head of the rarefaction where the $\partial^3 u/\partial x^3$ term is also negative, because fluid passing through this region passes into a stable region before an appreciable instability can develop.

Case (18d) was chosen to change the sign of the $\partial u/\partial x$ term. The instability that started at the foot of the rarefaction in cases (18b), (18c) has been eliminated, but there is a new instability, see Fig. 5. This instability is governed by the $\partial^2 u/\partial x^2$ term, since it grows only through the first half of the rarefaction, where $\partial^2 u/\partial x^2$ is positive, and then damps through the remainder of the rarefaction, where $\partial^2 u/\partial x^2$ is negative. The $\partial^3 u/\partial x^3$ term is seen to retard the growth of the instability at the head of the rarefaction. Apparently, the $\partial u/\partial x$ term, although positive, is too small to completely stabilize the rarefaction zone.

If difference scheme (18d) is modified to have mass fluxes evaluated at time step n instead of $(n + 1)$, the sign of the δt term in (19d) is changed. This modification yields results similar to those in Fig. 5, except for an additional instability growing in the constant-velocity region, because of the negative δt diffusion term.

Three additional comments should be made. First, the usual Fourier stability analysis cannot predict the observed instabilities, because it requires a linearization that neglects terms involving derivatives of u times derivatives of ρ. Although a linear Fourier analysis does say (18b) is unstable, because of the negative $u^2\delta t/2$ term in (19b), it does not detect the real source of instability.

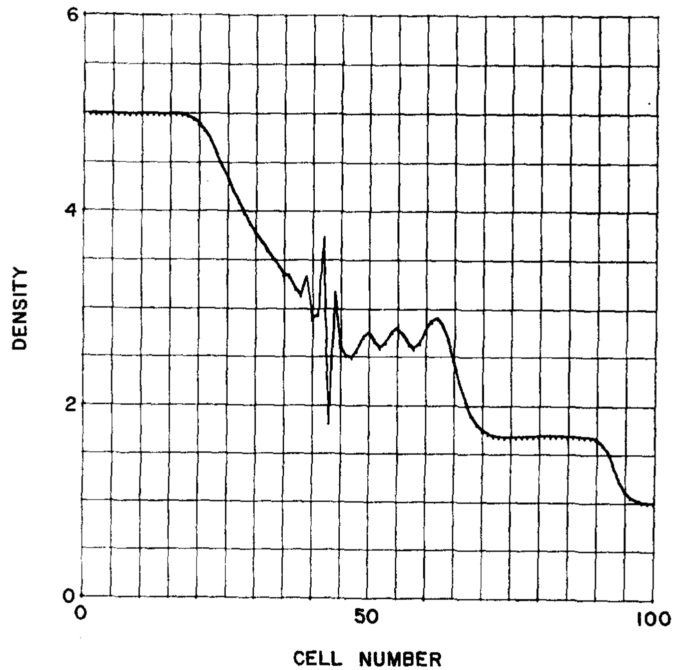

FIG. 4. A plot of density vs cell number at $t = 30.00$, as calculated by the centered-difference advanced-time method (18c).

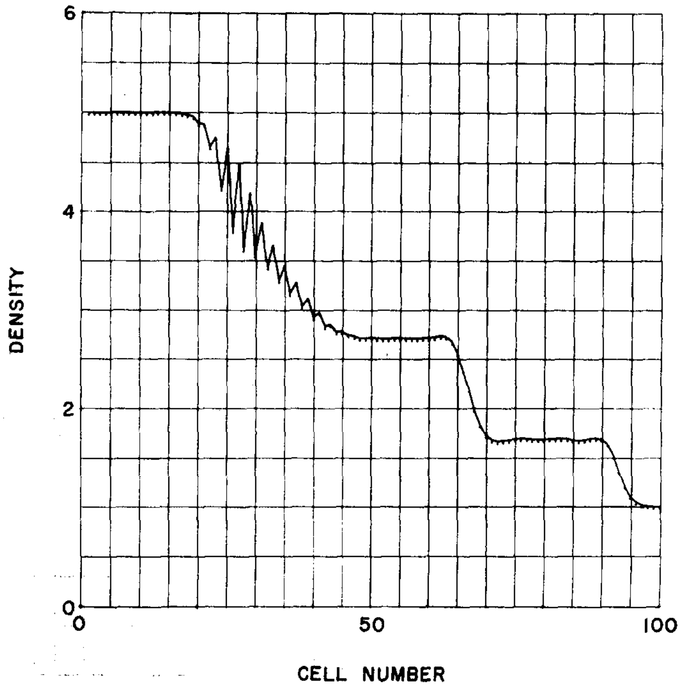

FIG. 5. A plot of density vs cell number at $t = 30.00$, as calculated by the second-order donor-cell method (18d).

Second, the effects of terms up to order δx^4 are easy to detect, because the discontinuous initial condition produces large velocity gradients at early times. If calculations are started with the stable donor-cell method and then a change is made to another difference approximation, (18b)–(18d), the instabilities can be weakened or completely eliminated depending on the length of time the donor-cell method is used.

Third, the instabilities in the rarefaction region of Figs. 3–5 appear to maintain a stationary amplitude. This phenomenon is not understood, but is probably connected with nonlinear effects produced by the instability itself. A study related to this has been made by Daly [6].

IV. Two-Dimensional Fluid Dynamics

As a final example, consider a calculation of the flow of water under a sluice gate. Figure 6 shows the computed configuration of fluid at $t = 0.280$ sec, as obtained

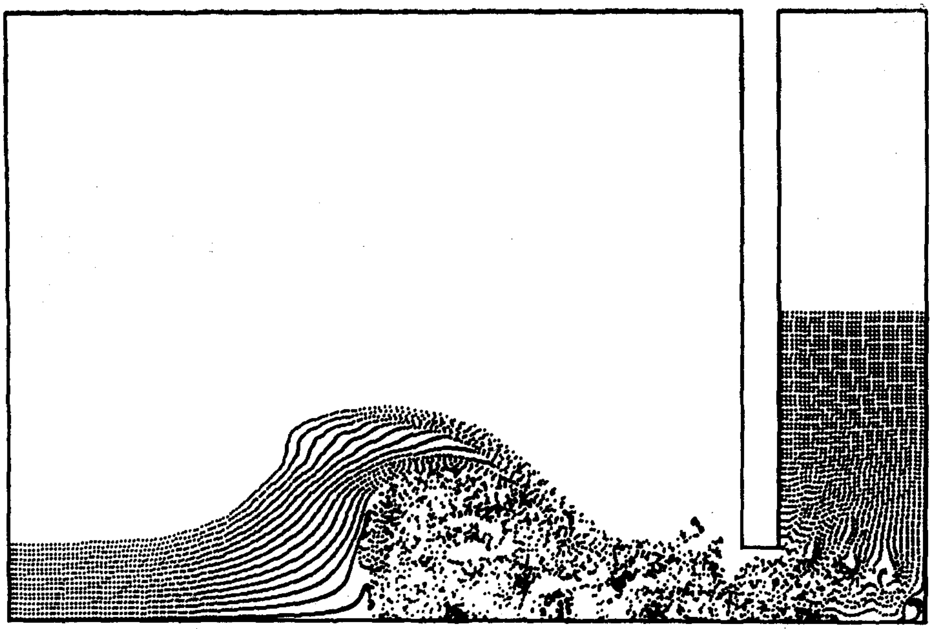

FIG. 6. Marker particle configuration, for calculation of water flowing under sluice gate, at $t = 0.280$ sec. Calculation was performed with $\delta t = 0.002$ sec.

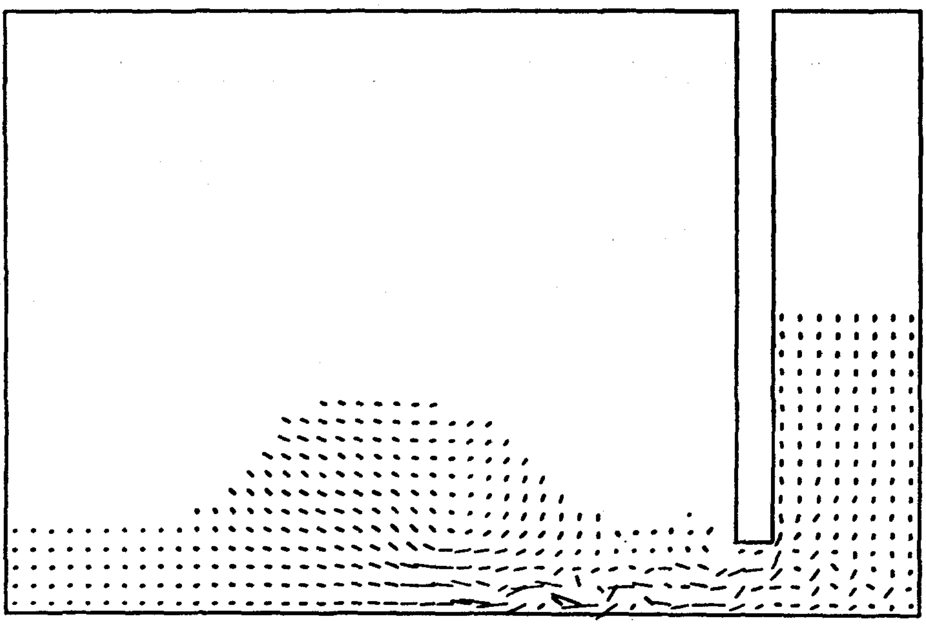

FIG. 7. Velocity vector plot, at $t = 0.280$ sec, corresponding to fluid configuration shown in Fig. 6.

by the MAC technique [2]. For this problem $\delta t = 0.002$ sec, $\delta x = \delta y = 0.03125$ ft, the kinematic viscosity coefficient is $\nu = 0.003$ ft^2/sec, and the acceleration of gravity is $g = 32.2$ ft/sec^2. A velocity vector plot, also at $t = 0.280$ sec, is shown in Fig. 7. Velocity vectors are drawn out from the center of each computing cell with a magnitude and direction characterizing the cell velocity.

An instability is developing in the lower right-hand portion of the fluid. The fluid preparing to flow under the sluice gate is unstable, it stabilizes as it passes under the gate, again becomes unstable as it "jets" under the right-hand-side of the surge wave, and then is finally stabilized under the left-hand-side of the surge wave.

Figure 8 shows the same calculation repeated with δt reduced by a factor of 20 ($\delta t = 0.0001$). The instability under the surge wave has been eliminated, but little change is seen behind the sluice gate.

These results can be readily understood in terms of truncation errors in the MAC difference equations. Keeping only the diffusion-like truncation errors, to order δt and δx^2, the horizontal and vertical velocity equations are, respectively,

$$\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) &= \left(\nu - \frac{\delta t}{2}u^2 - \frac{\delta x^2}{2}\frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x^2} \\ &\quad + \left(\nu - \frac{\delta t}{2}v^2 - \frac{\delta y^2}{4}\frac{\partial v}{\partial y}\right)\frac{\partial^2 u}{\partial y^2}, \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial}{\partial y}\left(\frac{p}{\rho}\right) + g &= \left(\nu - \frac{\delta t}{2}u^2 - \frac{\delta x^2}{4}\frac{\partial u}{\partial x}\right)\frac{\partial^2 v}{\partial x^2} \\ &\quad + \left(\nu - \frac{\delta t}{2}v^2 - \frac{\delta y^2}{2}\frac{\partial v}{\partial y}\right)\frac{\partial^2 v}{\partial y^2}.\end{aligned} \tag{20}$$

The incompressibility condition is

$$\partial u/\partial x + \partial v/\partial y = 0. \tag{21}$$

First, consider the flow region behind the sluice gate. A check of the velocity field in this region, before the instability sets in, reveals that $\partial u/\partial x$ is approximately equal to 16.0 sec^{-1} and the flow speed varies about unity. This means both the $\delta t = 0.002$ sec and $\delta t = 0.0001$ sec cases have a negative diffusion coefficient for $\partial^2 u/\partial x^2$ or $\partial^2 v/\partial x^2$, and a positive diffusion coefficient for $\partial^2 u/\partial y^2$ or $\partial^2 v/\partial y^2$. Without the space error proportional to δx^2, both coefficients would be positive and Eqs. (20) would have stable solutions. With the δx^2 terms we can explain why the instability occurs, why it does not vanish with reduced δt, and why it occurs with respect to the x coordinate and not the y coordinate.

Now, consider the region under the surge wave. The back of the wave (right-hand side) consists of a large eddy. Before an appreciable instability has developed,

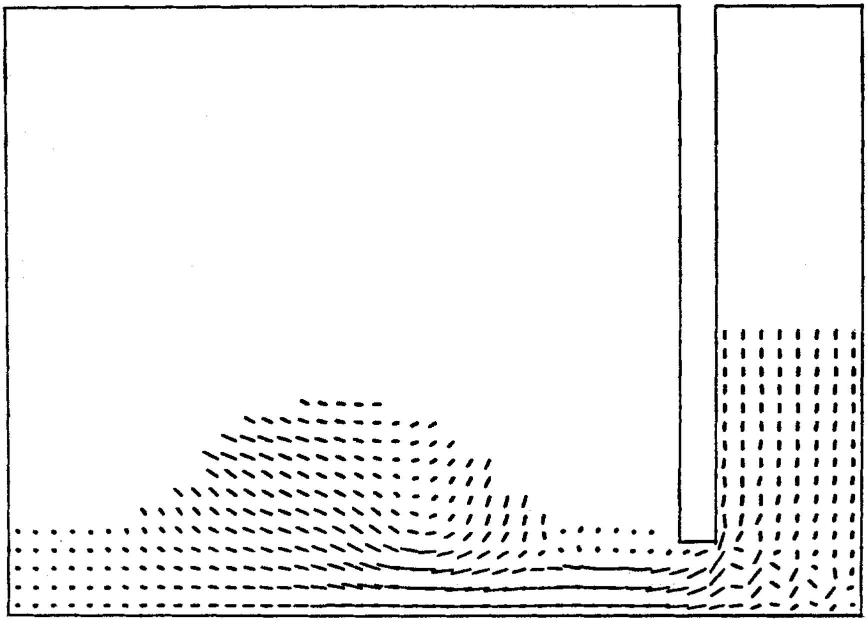

FIG. 8. Velocity vector plot, at $t = 0.280$ sec. Calculation was performed with $\delta t = 0.0001$ sec.

the flow from the sluice gate jets under this eddy with an average speed of approximately 6 ft/sec. Under the right side of the eddy $\partial v/\partial y$ is approximately $-29\ \text{sec}^{-1}$ and under the left side it is approximately $+20\ \text{sec}^{-1}$. In the $\delta t = 0.002$ sec case the δt diffusion term is large, making the jet region highly unstable. Since the δt diffusion term is detected by a linear stability analysis this is an example of linear instability.

In the $\delta t = 0.0001$ sec case, however, the jet is stable in the linear sense, but the coefficient of $\partial^2 u/\partial x^2$ or $\partial^2 v/\partial x^2$ is negative in the region of large negative $\partial v/\partial y$ values under the right side of the eddy. No instability is observed probably because the fluid passes through the unstable zone too quickly for an instability to develop appreciably. A similar situation occurred in example (18c) where the head of the rarefaction appeared stable, but should have been unstable.

As a useful *rule-of-thumb* the MAC method is considered stable if ν is greater than $1/2\ \delta t\ u^2$, where u is the average maximum fluid speed, and if ν is greater than $1/2\ \delta x^2(\partial u/\partial x)$, where $\partial u/\partial x$ is the average maximum velocity gradient in the direction of flow. The first condition is needed for linear stability. The second condition is a nonlinear stability requirement.

V. Concluding Remarks

The purpose of this paper is to present a technique that we have found very useful. In addition to the examples mentioned here, the truncation error method for determining computational stability has been tested on a variety of linear equations with uniformly good results. It has also been successfully applied to additional one-dimensional fluid calculations, and to other examples of MAC instabilities [7].

In addition to difference approximations (18a)–(18d) there is another approximation that is particularly interesting from the standpoint of truncation errors. The approximation is referred to as ZIP type differencing. It can be used to approximate the convection of any quantity Q defined at the same mesh locations as the convecting velocity. To illustrate the ZIP method, consider a one-dimensional convection term that is differenced about the point $x = j\delta x$ according to

$$\frac{\partial Qu}{\partial x} \approx \frac{1}{\delta x}\left[(Qu)_{j+1/2} - (Qu)_{j-1/2}\right].$$

This approximation is conservative if the right boundary value of Qu in cell j is equal to the left boundary value of Qu in cell $j + 1$. The ZIP method is a conservative method that defines the boundary value of the product as

$$(Qu)_{j+1/2} \equiv \frac{1}{2}\left[Q_j u_{j+1} + Q_{j+1} u_j\right].$$

The advantage of ZIP differencing is that it introduces no truncation errors contributing to a diffusion of Q. Thus, if ZIP differencing is used for the mass convection term in Section III, the effective mass diffusion coefficient contains no space errors, and hence, no instabilities associated with these errors. There will be an instability associated with non-time centering in the equation (a negative δt diffusion coefficient) and this must be compensated for in some way.

Acknowledgments

This paper reflects the efforts of every member of Group T-3 at the Los Alamos Scientific Laboratory. It is a pleasure to acknowledge their participation, and to particularly thank T. D. Butler, B. J. Daly, R. A. Gentry, and F. H. Harlow for their contributions.

References

1. G. G. O'Brien, M. A. Hyman, and S. Kaplan, *J. Math. Phys.* **29**, 223 (1950).
2. F. H. Harlow and J. E. Welch, *Phys. Fluids* **8**, 2182 (1965).

3. R. Courant, K. O. Friedricks, and H. Lewy, *Math. Ann.* **100**, 32 (1928).
4. J. von Neumann and R. D. Richtmyer, *J. Appl. Phys.* **21**, 232 (1950).
5. R. A. Gentry, R. E. Martin, and B. J. Daly, *J. Comp. Phys.* **1**, 87 (1966).
6. B. J. Daly, *Math. Comp.* **17**, 346 (1963).
7. B. J. Daly and W. E. Pracht, (unpublished).
8. R. D. Richtmyer, "Difference Methods for Initial Value Problems." Interscience, New York, 1957.